

SPIN OSCILLATIONS IN STORAGE RINGS

Alexander Silenko

Institute of Nuclear Problems,
Belarusian State University,
Minsk, Belarus

E-mail: silenko@inp.minsk.by

1 INTRODUCTION

The goal of this investigation is obtaining general formulae for spin oscillations in storage rings. Coherent betatron oscillations (CBOs) consist in an oscillatory motion of particles. A change in the particle trajectory is followed by a change in the focusing field acting on a particle. As a result, the magnetic field in the particle rest frame oscillates. This circumstance results in a shift of the spin rotation frequency in the g-2 experiment and a vertical motion of spin in the EDM one. Perhaps, spin oscillations can also affect other experiments. Therefore, the problem is very important.

Coherent betatron oscillations



Oscillations of acting field



Spin oscillations



Change of
spin rotation
frequency

(g-2 experiment)



Vertical motion
of spin



?

We should take into account the particle motion in storage rings is finite and periodical. The spin motion strongly depends on the particle one. It is quite natural to describe the spin motion in the cylindrical coordinate system. However, the Bargmann-Michel-Telegdi (BMT) equation uses the Cartesian coordinates. In the cylindrical coordinates, the azimuth is defined by the particle motion. The transformation of the BMT equation to these coordinates should be performed with an allowance for oscillatory terms in the particle motion equation. This circumstance makes the transformation rather difficult.

Thus, we need to calculate the particle motion in the horizontal plane and to take this motion into account when transforming the BMT equation to the cylindrical coordinates.

2 GEOMETRICAL ASPECTS OF PARTICLE AND SPIN OSCILLATIONS

The particle motion equation is given by

$$\frac{d\mathbf{p}}{dt} = e(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}), \quad \boldsymbol{\beta} = \frac{\mathbf{v}}{c}.$$

It is convenient to use the unit vector, $\mathbf{n} = \mathbf{p}/p$, which defines the direction of particle motion. The particle motion equation takes the form

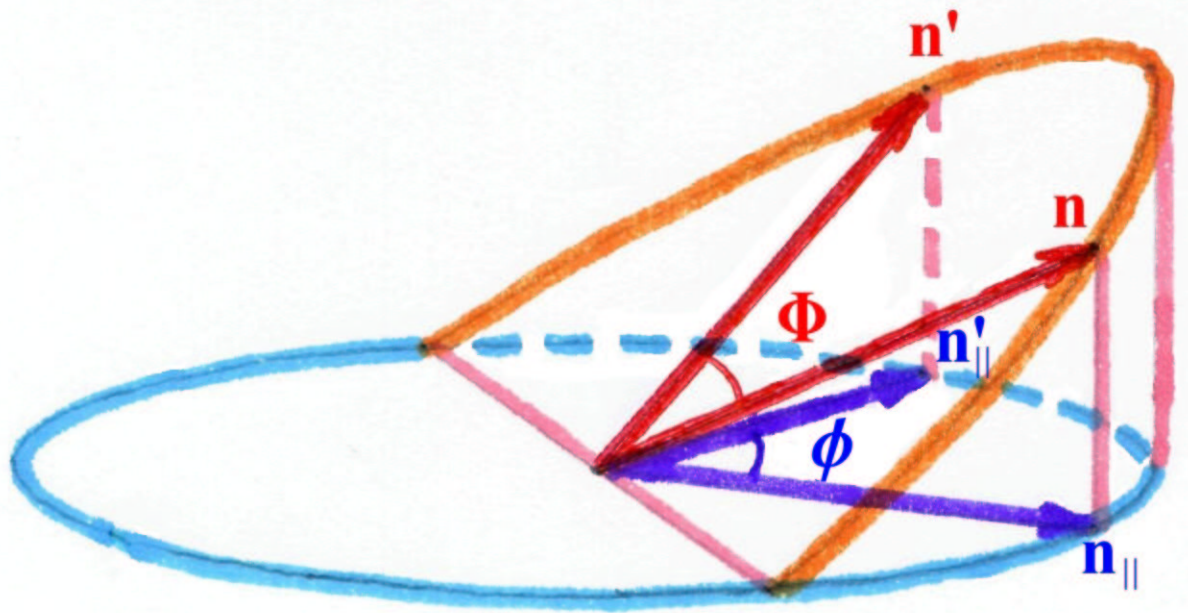
$$\frac{d\mathbf{n}}{dt} = \boldsymbol{\omega} \times \mathbf{n},$$
$$\boldsymbol{\omega} = -\frac{e}{\gamma m} \left(\mathbf{B} - \frac{\mathbf{n} \times \mathbf{E}}{\beta} \right),$$

where $\boldsymbol{\omega}$ is the angular velocity of the particle rotation.

Vertical and radial CBOs change the plans of particle and spin motion. The (pseudo)vectors of angular velocities become tilted. The angle Φ between two positions of any rotating vector \mathbf{n} does not equal to the angle ϕ between two corre-

sponding horizontal projections. Therefore, any additional field changes instantaneous frequencies of particle and spin motion.

Of course, the average frequencies of particle / spin motion in the tilted and horizontal planes are the same. Use of the horizontal plane for calculations is more convenient.



The small angle of the particle or spin rotation, ϕ , is defined by

$$\phi = \frac{(\mathbf{n}_{\parallel} \times \mathbf{n}'_{\parallel}) \cdot \mathbf{e}_z}{|\mathbf{n}_{\parallel}| \cdot |\mathbf{n}'_{\parallel}|} = \frac{(\mathbf{n}_{\parallel} \times \Delta \mathbf{n}_{\parallel}) \cdot \mathbf{e}_z}{|\mathbf{n}_{\parallel}|^2},$$

where $\Delta \mathbf{n}_{\parallel} = \mathbf{n}'_{\parallel} - \mathbf{n}_{\parallel}$. The instantaneous angular velocity of particle or spin rotation in the

horizontal plane is equal to

$$\dot{\phi} = \frac{(\mathbf{n}_{\parallel} \times \dot{\mathbf{n}}_{\parallel}) \cdot \mathbf{e}_z}{|\mathbf{n}_{\parallel}|^2}.$$

In the explicit form

$$\dot{\phi} = \omega_3 - \frac{(\omega_1 n_1 + \omega_2 n_2) n_3}{1 - n_3^2}. \quad (1)$$

$$1 \Rightarrow \mathbf{e}_r \text{ or } \mathbf{e}_x$$

$$2 \Rightarrow \mathbf{e}_{\phi} \text{ or } \mathbf{e}_y$$

$$3 \Rightarrow \mathbf{e}_z$$

This equation is exact.

3 CORRECTIONS TO THE PARTICLE MOTION IN THE HORIZONTAL PLANE

Usually, we can take into account only perturbations of particle orbit caused by the vertical and radial CBOs. In this case,

$$\begin{aligned} n_1 = n_r &= \frac{p_r}{p} = \rho_0 \sin(\omega_y t + \alpha), \\ n_3 = n_z &= \frac{p_z}{p} = \psi_0 \sin(\omega_p t + \beta). \end{aligned}$$

With an allowance for the orders of quantities

$$\begin{aligned} \omega_1 &\sim \psi_0, & \omega_2 &\sim \begin{cases} \rho_0^2 \\ \psi_0^2 \end{cases}, \\ n_1 &\sim \rho_0, & n_2 &\approx \pm 1, & n_3 &\sim \psi_0, \end{aligned}$$

we obtain that the second term in Eq. (1) is of the third order in the angular amplitudes of oscillations, ρ_0 and ψ_0 . Moreover, it oscillates and therefore it equals zero on the average. If we take into account only second-order terms in the

angular amplitudes, the second term in Eq. (1) is negligible. Finally,

$$\dot{\phi} = \omega_3 = -\frac{e}{\gamma m} \left(B_3 - \frac{(\mathbf{n} \times \mathbf{E})_3}{\beta} \right). \quad (2)$$

4 EQUATION OF SPIN MOTION IN STORAGE RINGS

In storage rings, the fields are determined in the cylindrical coordinate system and the spin motion is measured relatively the axes of this system. Therefore, we need to transform the general equation of spin motion to the cylindrical coordinates.

Let $\boldsymbol{\Omega}$ means the angular velocity of spin rotation in the Cartesian coordinates. The Bargmann-Michel-Telegdi equation which defines the spin motion has the form

$$\frac{d\mathbf{s}}{dt} = \boldsymbol{\Omega} \times \mathbf{s}, \quad \boldsymbol{\Omega} = -\frac{e}{m} \left[\left(a + \frac{1}{\gamma} \right) \mathbf{B} - \frac{a\gamma}{\gamma + 1} \boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{B}) - \left(a + \frac{1}{\gamma + 1} \right) (\boldsymbol{\beta} \times \mathbf{E}) \right],$$

where $a = (g - 2)/2$. We need to find the quantities

$$\frac{ds_r}{dt}, \quad \frac{ds_\phi}{dt}, \quad \text{and} \quad \frac{ds_z}{dt}.$$

In the xy -plane, the \mathbf{e}_r and \mathbf{e}_ϕ axes and the particle rotate with the same frequency. However, the particle can also move in the vertical plane, whereas the \mathbf{e}_r and \mathbf{e}_ϕ axes cannot. Consequently, these axes rotate with the instantaneous angular velocity

$$\boldsymbol{\omega}' = \dot{\phi} \mathbf{e}_z.$$

This property is confirmed by the simple calculation:

$$\begin{aligned} \frac{d\mathbf{e}_r}{dt} &= \frac{d}{dt} (\cos \phi \mathbf{e}_x + \sin \phi \mathbf{e}_y) = \dot{\phi} \mathbf{e}_\phi, \\ \frac{d\mathbf{e}_\phi}{dt} &= \frac{d}{dt} (-\sin \phi \mathbf{e}_x + \cos \phi \mathbf{e}_y) = -\dot{\phi} \mathbf{e}_r. \end{aligned}$$

The spin motion equation takes the form

$$\begin{aligned} \frac{ds_r}{dt} &= \frac{d\mathbf{s}}{dt} \cdot \mathbf{e}_r + \mathbf{s} \cdot \frac{d\mathbf{e}_r}{dt} = \mathbf{s} \cdot (\mathbf{e}_r \times \boldsymbol{\Omega}) + \dot{\phi} s_\phi, \\ \frac{ds_\phi}{dt} &= \frac{d\mathbf{s}}{dt} \cdot \mathbf{e}_\phi + \mathbf{s} \cdot \frac{d\mathbf{e}_\phi}{dt} = \mathbf{s} \cdot (\mathbf{e}_\phi \times \boldsymbol{\Omega}) - \dot{\phi} s_r, \\ \frac{ds_z}{dt} &= \mathbf{s} \cdot (\mathbf{e}_z \times \boldsymbol{\Omega}), \end{aligned}$$

or

$$\begin{aligned}
\frac{ds_r}{dt} &= -(\Omega_z - \dot{\phi})s_\phi + \Omega_\phi s_z, \\
\frac{ds_\phi}{dt} &= (\Omega_z - \dot{\phi})s_r - \Omega_r s_z, \\
\frac{ds_z}{dt} &= \Omega_r s_\phi - \Omega_\phi s_r.
\end{aligned} \tag{3}$$

We can formally introduce the Cartesian coordinate system with the axes $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ corresponding to the axes $\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z$, respectively. For such a system, Eq. (3) can be rewritten in the simple form:

$$\frac{d\mathbf{s}}{dt} = \boldsymbol{\omega}_a \times \mathbf{s}, \quad \boldsymbol{\omega}_a = \boldsymbol{\Omega} - \dot{\phi}\mathbf{e}_3. \tag{4}$$

The introduced coordinate system is purely mathematical and may not correspond to any real physical frame. The approximate conformity between this system and rotating frame takes place. Eq. (4) is exact. The quantity ω_a is the angular frequency of the $g-2$ precession.

Eqs. (1),(4) lead to the formula

$$\begin{aligned}
\boldsymbol{\omega}_a = & -\frac{e}{m} \left\{ a \mathbf{B} - \frac{a\gamma}{\gamma+1} \boldsymbol{\beta} (\boldsymbol{\beta} \cdot \mathbf{B}) \right. \\
& + \left(\frac{1}{\gamma^2 - 1} - a \right) (\boldsymbol{\beta} \times \mathbf{E}) \\
& \left. + \frac{1}{\gamma} \left(\mathbf{B}_{\parallel} - \frac{1}{\beta^2} (\boldsymbol{\beta} \times \mathbf{E})_{\parallel} \right) \right\} \\
& + \frac{(\omega_1 n_1 + \omega_2 n_2) n_3}{1 - n_3^2} \mathbf{e}_3,
\end{aligned} \tag{5}$$

where the sign \parallel means the components parallel to the $x_1 x_2$ -plane. This formula is exact. It describes the spin motion in an arbitrary storage ring with an allowance for the particle and spin oscillations and the EDM. As a rule, the last term is negligible.

Formulae for the electric dipole moment (EDM) can be obtained from the corresponding formulae for the anomalous magnetic moment by the substitution $\mathbf{B} \rightarrow \mathbf{E}$, $\mathbf{E} \rightarrow -\mathbf{B}$, $g-2 \rightarrow \eta$. The account of the particle EDM leads to the

following modification of the BMT equation:

$$\frac{d\mathbf{s}}{dt} = (\boldsymbol{\Omega} + \boldsymbol{\Omega}_{EDM}) \times \mathbf{s},$$

$$\boldsymbol{\Omega}_{EDM} = -\frac{e\eta}{2m} \left(\mathbf{E} - \frac{\gamma}{\gamma+1} \boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{E}) + \boldsymbol{\beta} \times \mathbf{B} \right), \quad (6)$$

where $\boldsymbol{\Omega}$ is defined by the BMT equation. The EDM does not affect the particle motion. As a rule, we can neglect term $\frac{\gamma}{\gamma+1} \boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{E})$.

After neglecting small terms, the equation for the angular frequency of the $g-2$ precession with an allowance for the EDM takes the form

$$\omega_a = -\frac{e}{m} \left\{ a\mathbf{B} - \frac{a\gamma}{\gamma+1} \boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{B}) \right. \\ \left. + \left(\frac{1}{\gamma^2 - 1} - a \right) (\boldsymbol{\beta} \times \mathbf{E}) \right. \\ \left. + \frac{1}{\gamma} \left[\mathbf{B}_{\parallel} - \frac{1}{\beta^2} (\boldsymbol{\beta} \times \mathbf{E})_{\parallel} \right] \right. \\ \left. + \frac{\eta}{2} (\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}) \right\}. \quad (7)$$

This formula makes it possible to take into account the spin oscillations.

5 SPIN OSCILLATIONS IN THE g-2 EXPERIMENT

In the g-2 experiment, the correction for the vertical CBO (pitch) is about 0.2 ppm. This effect has been described by F. Farley

[F.J.M. Farley, *Phys. Lett. B* **42**, 66 (1972).]

The result has been confirmed by J. Field and G. Fiorentini

[J.H. Field and G. Fiorentini, *Nuovo Cim. A* **21**, 297 (1974)]

and by computer simulations.

The horizontal CBO (yaw) does not give any significant corrections.

With the above formulae, the theory of spin oscillations in the g-2 experiment can be developed in the very general form. The spin motion perturbed by the vertical CBO is described by

the equation

$$\begin{aligned} \frac{d\mathbf{s}}{dt} = & \{a_0 + a_3 \cos [2(\omega_p t + \phi_p)]\} (\mathbf{e}_3 \times \mathbf{s}) \\ & + a_2 \cos (\omega_p t + \phi_p) (\mathbf{e}_2 \times \mathbf{s}) \\ & + a_1 \sin (\omega_p t + \phi_p) (\mathbf{e}_1 \times \mathbf{s}), \end{aligned}$$

where a_1 and a_2 are first-order quantities and a_3 is a second-order quantity in the angular amplitude of pitch, ψ_0 . The quantity ω_p is the angular frequency of pitch.

In the explicit form,

$$\begin{aligned} \frac{ds_1}{dt} = & - \{a_0 + a_3 \cos [2(\omega_p t + \phi_p)]\} s_2 \\ & + a_2 \cos (\omega_p t + \phi_p) s_3, \\ \frac{ds_2}{dt} = & \{a_0 + a_3 \cos [2(\omega_p t + \phi_p)]\} s_1 \\ & - a_1 \sin (\omega_p t + \phi_p) s_3, \\ \frac{ds_3}{dt} = & -a_2 \cos (\omega_p t + \phi_p) s_1 \\ & + a_1 \sin (\omega_p t + \phi_p) s_2. \end{aligned}$$

This equation has the exact solution

$$\begin{aligned}
s_1 &= s_{\parallel}^{(0)} \cos(a_0 t + \phi_0) \\
&+ \frac{a_0 a_1 - a_2 \omega_p}{a_0^2 - \omega_p^2} s_3^{(0)} \sin(\omega_p t + \phi_p), \\
s_2 &= s_{\parallel}^{(0)} \sin(a_0 t + \phi_0) \\
&+ \frac{a_0 a_2 - a_1 \omega_p}{a_0^2 - \omega_p^2} s_3^{(0)} \cos(\omega_p t + \phi_p), \\
s_3 &= s_3^{(0)} \\
&- \left[\frac{a_0 a_2 - a_1 \omega_p}{a_0^2 - \omega_p^2} \sin(a_0 t + \phi_0) \cos(\omega_p t + \phi_p) \right. \\
&\left. + \frac{a_0 a_1 - a_2 \omega_p}{a_0^2 - \omega_p^2} \cos(a_0 t + \phi_0) \sin(\omega_p t + \phi_p) \right] s_{\parallel}^{(0)}.
\end{aligned}$$

Approximately,

$$\begin{aligned}
s_1 &= s_{\parallel}^{(0)} \cos(a_0 t + \phi_0), \\
s_2 &= s_{\parallel}^{(0)} \sin(a_0 t + \phi_0), \\
s_3 &= s_3^{(0)},
\end{aligned}$$

where ϕ_0 is an initial phase. The spin vector is normalized to unit:

$$\left(s_{\parallel}^{(0)} \right)^2 + \left(s_3^{(0)} \right)^2 = 1.$$

The result can be substituted into the above formulae. If only second-order terms in the pitch amplitude, ψ_0 , are taken into account, averaging results in the expression

$$\omega_a = \langle \dot{\phi} \rangle = a_0 + \frac{a_0(a_1^2 + a_2^2) - 2a_1a_2\omega_p}{4(a_0^2 - \omega_p^2)} + \frac{a_0(a_1^2 - a_2^2)}{4(a_0^2 - \omega_p^2)} \langle \cos 2(a_0t + \phi_0) \rangle \cdot \frac{1 + (s_3^{(0)})^2}{1 - (s_3^{(0)})^2}.$$

In preceding works, both electric and magnetic focusing have been considered. The coefficients a_i ($i = 0, 1, 2, 3$) are equal to

$$\begin{aligned} a_0 &= \lambda\omega_0 \left(1 - \frac{\gamma - 1}{2\gamma} \psi_0^2 \right), \\ a_1 &= -\omega_0 \frac{\gamma - 1}{\gamma} \psi_0, \quad a_2 = -\lambda f \omega_p \psi_0, \\ a_3 &= \lambda\omega_0 \frac{\gamma - 1}{2\gamma} \psi_0^2, \end{aligned}$$

where λ equals 1 and -1 for negative and positive muons, respectively. The factor f is

$$f = 1 + a\gamma - \frac{1 + a}{\gamma} = 1 + a\beta^2\gamma - \frac{1}{\gamma}$$

and

$$f = 1 + a\gamma$$

for electric and magnetic focusing, respectively.

The g-2 frequency equals

$$\begin{aligned} \omega_a &= \omega_0(1 - C), \\ C &= \frac{1}{4}\psi_0^2 \left[1 - \frac{\omega_0^2}{\gamma^2(\omega_0^2 - \omega_p^2)} \right. \\ &\quad \left. - \frac{\omega_p^2(f - 1)(f - 1 + 2/\gamma)}{\omega_0^2 - \omega_p^2} \right. \\ &\quad \left. - D < \cos [2(\omega_0 t + \phi_0)] > \cdot \frac{1 + (s_3^{(0)})^2}{1 - (s_3^{(0)})^2} \right], \\ D &= \frac{(\gamma - 1)^2\omega_0^2 - f^2\gamma^2\omega_p^2}{\gamma^2(\omega_0^2 - \omega_p^2)}. \end{aligned} \quad (8)$$

Of course, the average value of the last term oscillating with the frequency $2\omega_0$ is zero. Therefore, formula (8) is in the best agreement with the result found by F. Farley

[F.J.M. Farley, Phys. Lett. B **42**, 66 (1972).]

In this reference, only the case $s_3^{(0)} = 0$ has been considered.

However, it is possible to include the oscillatory term in a fitting process instead of its elimination. In the current g-2 experiment, $f = 1$, $s_3^{(0)} = 0$, $\gamma \gg 1$, ϕ_0 equals 0 or π , and formula (8) takes the form

$$\omega_a = \omega_0(1-C), \quad C = \frac{1}{4}\psi_0^2 [1 - \langle \cos(2\omega_0 t) \rangle].$$

This formula shows the inclusion of the oscillatory term in a fitting process is not difficult.

6 SPIN OSCILLATIONS IN THE EDM EXPERIMENT

In the EDM experiment, the spin motion in the horizontal plane is strongly restricted with the radial electric field. In this case, the spin rotation about the radial axis becomes very important because this rotation imitates the EDM effect. For the calculation of the correction caused by the vertical CBO, the designation of the axes needs to be changed:

$$\begin{aligned} 1 &\Rightarrow \mathbf{e}_\phi \\ 2 &\Rightarrow \mathbf{e}_z \\ 3 &\Rightarrow \mathbf{e}_r \end{aligned}$$

In the considered case, the vertical CBO leads to the spin motion described by the equation

$$\begin{aligned} \frac{d\mathbf{s}}{dt} = & \{a_0 + a_3 \cos(\omega_p t + \phi_p)\} (\mathbf{e}_3 \times \mathbf{s}) \\ & + a_1 \sin(\omega_p t + \phi_p) (\mathbf{e}_1 \times \mathbf{s}), \end{aligned}$$

where a_1 and a_3 are first-order quantities in the angular amplitude of pitch, ψ_0 .

This equation also has the exact solution:

$$\begin{aligned} \omega_a = & a_0 + \frac{a_0 a_1^2}{4(a_0^2 - \omega_p^2)} \left[1 \right. \\ & \left. + \frac{1 + \left(s_3^{(0)} \right)^2}{1 - \left(s_3^{(0)} \right)^2} \right] \\ & + \frac{a_1 a_3}{2\omega_p} \cdot \frac{s_3^{(0)}}{s_{\parallel}^{(0)}} \langle \sin(a_0 t + \phi_0) \rangle . \end{aligned}$$

In the EDM experiment

$$\begin{aligned} \omega_0 = |a_0|, \quad a_1 = a\omega_c\psi_0, \quad a_3 = \lambda f\omega_p\psi_0, \\ \phi_0 = 0 \text{ or } \pi, \quad s_3^{(0)} = 0, \quad a = \frac{g-2}{2}, \\ \langle \cos[2(a_0 t + \phi_0)] \rangle = 1, \\ \langle \sin(a_0 t + \phi_0) \rangle = 0, \end{aligned}$$

where ω_c is the cyclotron frequency and $f = 1 + a\gamma$.

The equation for the angular frequency takes the form:

$$\omega_a = |a_0| \left[1 + \frac{a_1^2}{2(a_0^2 - \omega_p^2)} \right] \approx \omega_0 \left(1 - \frac{a^2 \omega_c^2}{2\omega_p^2} \psi_0^2 \right).$$

The yaw correction for EDM experiment is non-zero. In this case, the spin motion equation is similar to the corresponding equation for the g-2 experiment.

The equation for the angular frequency has the form

$$\omega_a = \omega_0 \left[1 + a(\gamma - 1)\gamma \left(\frac{\omega_p^2}{2\omega_y^2} - \frac{1}{g} \right) \rho_0^2 \right].$$

Vertical and radial CBOs give only small corrections to the systematical errors caused by the vertical electric field. These corrections can be neglected.

7 SUMMARY

- The angle between two vectors in a tilted plane does not equal to the angle between the projections of these vectors on the horizontal plane. This geometrical effect leads to the dependence on spin rotation frequency on beam and spin oscillations.
- The exact formula describing the dependence of the particle rotation frequency on the particle orbit perturbations is derived. For the g-2 and EDM experiments, the corrections are negligible.
- The exact equation of spin motion is found in the cylindrical coordinate system. This equation describes the spin motion relatively detectors.
- The formula for the frequency of g-2 precession is in the best agreement with

the result found by F. Farley. With respect to this result, the found formula contains the additional oscillatory term that can be used for fitting.

- The influence of spin oscillations on the spin dynamics in the EDM experiment is negligible.