Classical spinning elementary particles, *zitterbewegung* and dipole structure.

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Abstract

The concept of elementary particle rests on the idea that it is a physical system with no excited states, so that all possible kinematical states of the particle are just kinematical modifications of any one of them. In this way instead of describing the particle attributes it amounts to describe the collection of consecutive inertial observers who describe the particle in the same kinematical state. The kinematical state space of an elementary particle is a homogeneous space of the kinematical group. By considering the largest homogeneous spaces of both, Galilei and Poincaré groups, it is shown how the spin structure is related to the different degrees of freedom. After a review of the main features of the structure of the Dirac electron, it is analyzed how the classical spin and dipole structure and *zitterbewegung* are related to the quantum features of the particle.

Definition: An **elementary particle** is a mechanical system without excited states. All its kinematical states are kinematical modifications of any one of them.

This implies that if the state of the particle changes it is always possible to find a new inertial observer who describes the elementary particle in the same kinematical state as before.

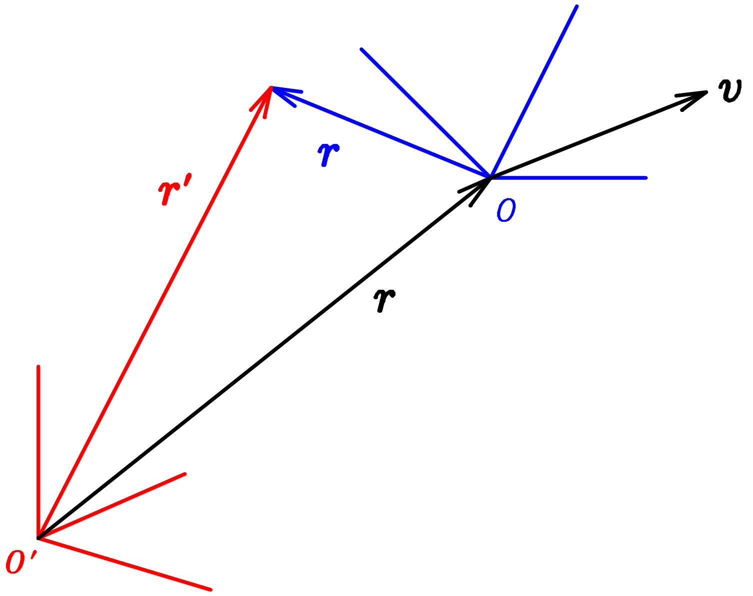
When the particle changes its state x into $x + dx$ at the next instant, this implies that the new values of the kinematical variables $x + dx$ will be obtained from the preceeding ones x by an infinitesimal transformation δg of the kinematical group G of space-time symmetries.

Corollary: The manifold spanned by the kinematical variables is necessarily a **homogeneous space** of the kinematical group G.

The evolution of an elementary particle is thus reduced to the analysis of the updated consecutive inertial observers who describe the particle in the same state.

Galilei group

$$\begin{aligned} t' &= t + t, \\ \boldsymbol{r}' &= \mathcal{R}(\boldsymbol{\alpha})\boldsymbol{r} + \boldsymbol{v}t + \boldsymbol{r} \end{aligned}$$



where the black variables t, $\boldsymbol{r}$, $\boldsymbol{v}$ and $\mathcal{R}(\boldsymbol{\alpha})$ are the relative group parameters between inertial observers O and O'. The meaning of these parameters is

- t and $\boldsymbol{r}$ correspond to the time and position of the origin of O frame as measured by O', considered as the event $t = 0$, $\boldsymbol{r} = 0$ in O frame.

- $\boldsymbol{v}$ is the velocity of the origin of O frame as measured by O'.

- $\mathcal{R}(\boldsymbol{\alpha})$ is the orientation of O cartesian frame with respecto to O' frame, and is expressed in terms of the rotation matrix $\mathcal{R}(\boldsymbol{\alpha})$ to be applied to O' frame to match with O's.

It therefore implies that O' observer describes any other inertial observer by giving a time, position, velocity and orientation of the origin of the other inertial observer. This is the maximum information we can handle for describing the kinematical state of an elementary particle.

Generalized Lagrangian formalism

Any Lagrangian system of n degrees of freedom q_i which depends on time t, on the q_i and their time derivatives up to a finite order k, $q_i^{(k)}$,

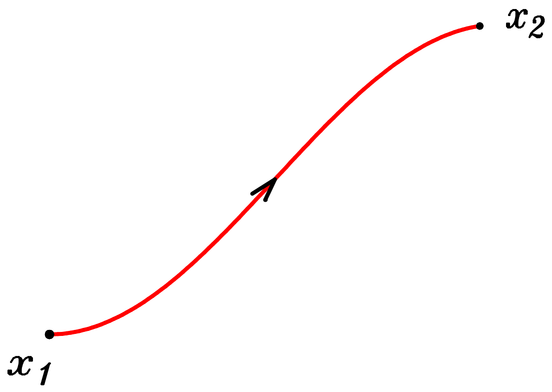
$$\int_{t_1}^{t_2} L(t, q_i, \ldots, q_i^{(k-1)}, q_i^{(k)})\, dt = \int_{\tau_1}^{\tau_2} L(x, \dot{x}) d\tau$$

the kinematical Lagrangian space is spanned by the variables

$$x \equiv (t, q_i, \ldots, q_i^{(k-1)}),$$

so that if we describe the evolution in terms of some arbitrary evolution parameter τ can always be rewritten in terms of the kinematical variables and their next order $\tau-$derivative.

$$L(x,\dot{x}) = \frac{\partial L}{\partial \dot{x}_j}\dot{x}_j = P_j(x,\dot{x})\dot{x}_j.$$



Function $L(x,\dot{x})$ is a homogeneous function of first degree of the derivatives $\dot{x}_j$, and thus:

The $P_j(x,\dot{x})$ are homogeneous functions of zero degree of the derivatives $\dot{x}_j$.

Newtonian point-particle

The Newtonian point-particle is an elementary particle according to this definition. Its only kinematical variables are time t and position $\boldsymbol{r}$. When describing the evolution in terms of some arbitrary evolution parameter τ

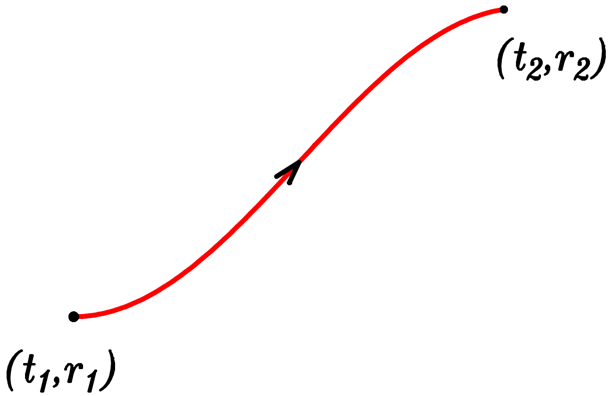
$$\int_{t_1}^{t_2} \frac{m}{2} \left(\frac{d\boldsymbol{r}}{dt}\right)^2 dt = \int_{\tau_1}^{\tau_2} \frac{m}{2} \frac{\dot{\boldsymbol{r}}^2}{\dot{t}^2} \dot{t}\, d\tau = \int_{\tau_1}^{\tau_2} \frac{m}{2} \frac{\dot{\boldsymbol{r}}^2}{\dot{t}} d\tau$$

Its kinematical variables are t and $\boldsymbol{r}$, and therefore the Lagrangian which describes this system $L(t, \boldsymbol{r}, \dot{t}, \dot{\boldsymbol{r}})$ will be in general a function of the kinematical variables and their next order $\tau-$derivative, so that

$$L_0 = \frac{m}{2} \frac{\dot{\boldsymbol{r}}^2}{\dot{t}}$$

is a homogeneous function of first degree of the $\tau-$derivatives of the kinematical variables.

$$L_0 = \frac{m}{2}\frac{\dot{\boldsymbol{r}}^2}{\dot{t}} = \dot{t}T + \dot{\boldsymbol{r}}\cdot\boldsymbol{R},$$



$$T = \frac{\partial L_0}{\partial \dot{t}} = -\frac{m}{2}\frac{\dot{\boldsymbol{r}}^2}{\dot{t}^2}, \qquad \boldsymbol{R} = \frac{\partial L_0}{\partial \dot{\boldsymbol{r}}} = m\frac{\dot{\boldsymbol{r}}}{\dot{t}}.$$

Relativistic point-particle

The relativistic point-particle is also described by a Lagrangian which is a homogeneous function of first degree of the $\tau-$derivatives of the kinematical variables.

$$\int_{t_1}^{t_2} -mc^2\sqrt{1-\frac{1}{c^2}\left(\frac{d\boldsymbol{r}}{dt}\right)^2}\,dt = \int_{\tau_1}^{\tau_2} -mc\sqrt{c^2\dot{t}^2-\dot{\boldsymbol{r}}^2}\,d\tau$$

$$L_0 = -mc\sqrt{c^2\dot{t}^2-\dot{\boldsymbol{r}}^2} = \dot{t}T + \dot{\boldsymbol{r}}\cdot\boldsymbol{R},$$

$$T = \frac{\partial L_0}{\partial \dot{t}} = -\frac{mc^3\dot{t}}{\sqrt{c^2\dot{t}^2-\dot{\boldsymbol{r}}^2}}, \qquad \boldsymbol{R} = \frac{\partial L_0}{\partial \dot{\boldsymbol{r}}} = \frac{mc\dot{\boldsymbol{r}}}{\sqrt{c^2\dot{t}^2-\dot{\boldsymbol{r}}^2}}.$$

The action of the system

If at instant τ we describe an inertial observer by the variables

$$\tau \quad \longrightarrow \quad (t, \boldsymbol{r}, \boldsymbol{v}, \boldsymbol{\alpha})$$

at instant $\tau + \delta\tau$ they will be

$$\tau + \delta\tau \quad \longrightarrow \quad (t + \delta t, \boldsymbol{r} + \delta\boldsymbol{r}, \boldsymbol{v} + \delta\boldsymbol{v}, \boldsymbol{\alpha} + \delta\boldsymbol{\alpha})$$

so that the infinitesimal change of the action of the system will be proportional to this infinitesimal change

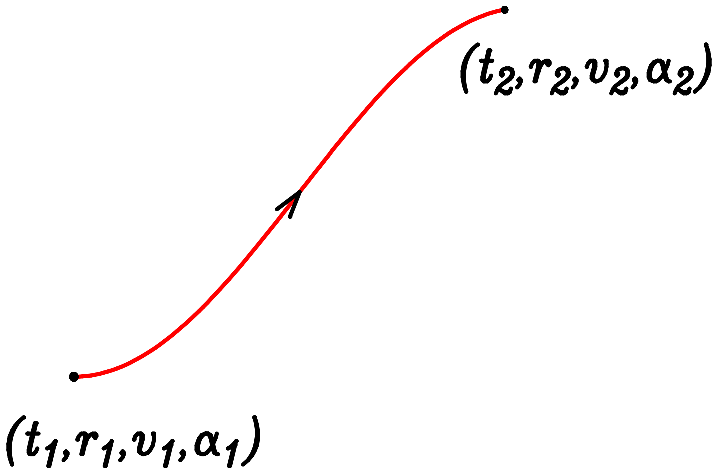
$$T\delta t + \boldsymbol{R} \cdot \delta\boldsymbol{r} + \boldsymbol{V} \cdot \delta\boldsymbol{v} + \boldsymbol{W} \cdot \delta\boldsymbol{\alpha} = P_j \delta x_j$$

$$(T\dot{t} + \boldsymbol{R} \cdot \dot{\boldsymbol{r}} + \boldsymbol{V} \cdot \dot{\boldsymbol{v}} + \boldsymbol{W} \cdot \boldsymbol{\omega})\delta\tau = L\delta\tau = (P_j \dot{x}_j)\delta\tau$$

and represents the expansion of the homogeneous Lagrangian in terms of the derivatives of the kinematical variables.

The most general elementary particle

$$L_0 = \dot{t}T + \dot{\boldsymbol{r}} \cdot \boldsymbol{R} + \dot{\boldsymbol{v}} \cdot \boldsymbol{V} + \boldsymbol{\omega} \cdot \boldsymbol{W},$$



$$T = \frac{\partial L_0}{\partial \dot{t}}, \qquad \boldsymbol{R} = \frac{\partial L_0}{\partial \dot{\boldsymbol{r}}}, \qquad \boldsymbol{V} = \frac{\partial L_0}{\partial \dot{\boldsymbol{v}}}, \qquad \boldsymbol{W} = \frac{\partial L_0}{\partial \boldsymbol{\omega}}.$$

Noether's theorem: Constants of the motion

Energy

$$H = -T - \boldsymbol{v} \cdot \frac{d\boldsymbol{V}}{dt},$$

Linear momentum

$$\boldsymbol{P} = \boldsymbol{R} + \frac{d\boldsymbol{V}}{dt},$$

Kinematical momentum

$$\boldsymbol{K} = \frac{H}{c^2}\boldsymbol{r} - \boldsymbol{P}t - \boldsymbol{S} \times \frac{\boldsymbol{v}}{c^2}$$

Angular momentum

$$\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{P} + \boldsymbol{v} \times \boldsymbol{V} + \boldsymbol{W} = \boldsymbol{r} \times \boldsymbol{P} + \boldsymbol{S}$$

$$\boldsymbol{S} = \boldsymbol{v} \times \boldsymbol{V} + \boldsymbol{W},$$

The time derivative of the constant $\boldsymbol{K}$ leads to

$$\frac{d\boldsymbol{K}}{dt} = 0 = \frac{H}{c^2}\boldsymbol{v} - \boldsymbol{P} - \frac{d}{dt}\left(\boldsymbol{S} \times \frac{\boldsymbol{v}}{c^2}\right) \quad \Rightarrow \quad \boldsymbol{P} = \frac{H}{c^2}\boldsymbol{v} - \frac{1}{c^2}\frac{d}{dt}\left(\boldsymbol{S} \times \boldsymbol{v}\right).$$

$\boldsymbol{P}$ and $\boldsymbol{v}$ are not collinear vectors.

If we define the position of a point $\boldsymbol{q}$ through

$$\frac{H}{c^2}\boldsymbol{q} = \frac{H}{c^2}\boldsymbol{r} - \boldsymbol{S} \times \frac{\boldsymbol{v}}{c^2}, \qquad \Longrightarrow \qquad \boldsymbol{q} = \boldsymbol{r} - \frac{1}{H}\boldsymbol{S} \times \boldsymbol{v},$$

then

$$\boldsymbol{K} = \frac{H}{c^2}\boldsymbol{q} - \boldsymbol{P}t, \quad \text{and thus} \quad \boldsymbol{P} = \frac{H}{c^2}\frac{d\boldsymbol{q}}{dt}.$$

Point $\boldsymbol{q}$ represents the position of the center of mass of the particle.

Whenever the system has a spin $\boldsymbol{S} \neq 0$, the linear momentum is not along the velocity $\boldsymbol{v}$ of point $\boldsymbol{r}$. Point $\boldsymbol{r}$ does not represent the position of the center of mass of the particle. It is the position of the charge.

The six conditions $\boldsymbol{P} = 0$ and $\boldsymbol{K} = 0$ imply $\boldsymbol{q} = 0$ and $d\boldsymbol{q}/dt = 0$. They define the class of center of mass observers.

The time derivative of the constant $\boldsymbol{J}$ leads to

$$\frac{d\boldsymbol{J}}{dt} = 0 = \boldsymbol{v} \times \boldsymbol{P} + \frac{d\boldsymbol{S}}{dt} \quad \Rightarrow \quad \frac{d\boldsymbol{S}}{dt} = \boldsymbol{P} \times \boldsymbol{v}.$$

The spin is only a constant of the motion for the center of mass observer.

Dirac analysis of the electron

Let point $\boldsymbol{r}$ be the position vector on which the four component Dirac's spinor $\psi(t, \boldsymbol{r})$ is defined. Dirac's Hamiltonian is

$$H = c\boldsymbol{P} \cdot \boldsymbol{\alpha} + \beta mc^2,$$

where α_i and β are Dirac's 4×4 matrices.

When computing the velocity of point $\boldsymbol{r}$, Dirac arrives at:

1. The velocity of point $\boldsymbol{r}$, $\boldsymbol{v} = i/\hbar[H, \boldsymbol{r}] = c\boldsymbol{\alpha}$, is expressed in terms of $\boldsymbol{\alpha}$ matrices and writes, *'... a measurement of a component of the velocity of a free electron is certain to lead to the result* $\pm c$. This conclusion is easily seen to hold also when there is a field present.'

2. The linear momentum does not have the direction of this velocity $\boldsymbol{v}$, but must be related to some average value of it: . . . *'the x_1 component of the velocity, $c\alpha_1$, consists of two parts, a constant part $c^2p_1H^{-1}$, connected with the momentum by the classical relativistic formula, and an oscillatory part, whose frequency is at least $2mc^2/h$, . . .'*.

3. About the position $\boldsymbol{r}$: *'The oscillatory part of x_1 is small, . . . , which is of order of magnitude $\hbar/mc$, . . .'*.

4. About the spin dynamics he obtains

$$\frac{d\boldsymbol{S}}{dt} = \boldsymbol{P} \times c\boldsymbol{\alpha} = \boldsymbol{P} \times \boldsymbol{v}.$$

A classical description of the electron

In the relativistic case we have three possible maximal, disjoint, homogeneous spaces of the Poincaré group, spanned by the ten variables $(t, \boldsymbol{r}, \boldsymbol{v}, \boldsymbol{\alpha})$, provided $v < c$, $v > c$ or $v = c$.

The very important manifold for the description of the classical photon and electron is the one with $v = c$, because it describes a system whose position vector $\boldsymbol{r}$, which is not the center of mass of the system, it is moving at the speed of light. The importance of this manifold is suggested by Dirac's previous analysis of the electron.In fact, it is the quantization of this system which leads to Dirac equation.

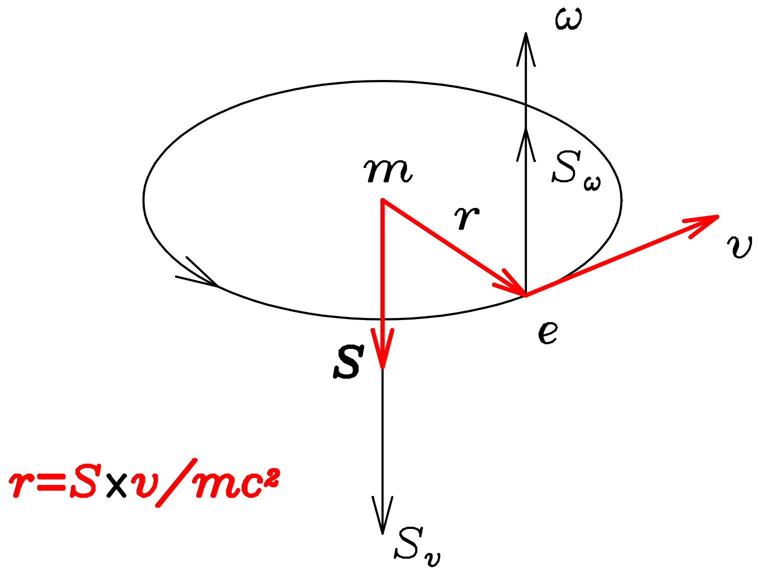
For the center of mass observer $\boldsymbol{K} = \boldsymbol{P} = 0$, the spin is a constant of the motion, $H = mc^2$ and thus

$$\boldsymbol{K} = \frac{H}{c^2}\boldsymbol{r} - \boldsymbol{P}t - \frac{1}{c^2}\boldsymbol{S} \times \boldsymbol{v} = 0, \qquad \Rightarrow \qquad \boldsymbol{r} = \frac{1}{mc^2}\boldsymbol{S} \times \boldsymbol{v},$$

so that point $\boldsymbol{r}$ is moving in circles, at the speed of light, on a plane orthogonal to the constant vector $\boldsymbol{S}$. Classical mechanics does not restrict the value of the constant spin S which can be any positive real number. Its true value will be uniquely fixed after quantization.

The radius of this circle is $R = S/mc$ and the angular velocity of this internal motion or *zitterbewegung* is $\omega = mc^2/S$.

Motion of the electron charge in the center of mass frame



$$\boldsymbol{r} = \boldsymbol{S} \times \boldsymbol{v} / mc^2$$

The ground state energy of this system when quantized,

$$R = S/mc, \quad \omega = mc^2/S, \quad \hbar\omega/2 = mc^2, \quad \Rightarrow \quad S = \hbar/2.$$

In the center of mass frame it is a system of three degrees of freedom. The coordinates x and y of the point $\boldsymbol{r}$ and the phase α of the rotation of the body frame. This phase is the same as the phase of the orbital motion. The motion is at a constant velocity c, then the system is reduced to a single degree of freedom system. It is a one-dimensional harmonic oscillator of frequency $\omega = mc^2/S$, without excited states.

The negative energy particle corresponds to the time reversed motion with the same spin $\boldsymbol{S}$.

Dirac equation

The classical expression that leads to Dirac equation when quantizing the system comes from $\boldsymbol{K}$.

$$\boldsymbol{K} = \frac{H}{c^2}\boldsymbol{r} - \boldsymbol{P}t - \boldsymbol{S} \times \frac{\boldsymbol{v}}{c^2}, \quad \Rightarrow \quad \frac{d\boldsymbol{K}}{dt} = 0 = \frac{H}{c^2}\boldsymbol{v} - \boldsymbol{P} - \frac{d}{dt}\left(\boldsymbol{S} \times \frac{\boldsymbol{v}}{c^2}\right)$$

$$H = \boldsymbol{P} \cdot \boldsymbol{v} + \frac{1}{c^2}\boldsymbol{S} \cdot \left(\frac{d\boldsymbol{v}}{dt} \times \boldsymbol{v}\right).$$

This is a linear relationship between H and $\boldsymbol{P}$, where the velocity $\boldsymbol{v}$ should be replaced by Dirac's velocity operator $c\boldsymbol{\alpha}$ and the last term corresponds to βmc^2 in terms of Dirac's β matrix. In the center of mass frame the absolute value of the acceleration is c^2/R, so that taking into account the value of R we get that this term reduces to $\pm mc^2$, the positive value for the particle and the negative one for the antiparticle.

Dirac dipole structure

And when analysing, in his original 1928 paper the interaction of the electron with an external electromagnetic field, he obtains two new interaction terms:

$$\frac{e\hbar}{2mc}\boldsymbol{\sigma}\cdot\boldsymbol{B}+\frac{ie\hbar}{2mc}\boldsymbol{\alpha}\cdot\boldsymbol{E}, \tag{1}$$

He says, *'The electron will therefore behave as though it has a magnetic moment* $(e\hbar/2mc)\,\boldsymbol{\sigma}$ *and an electric moment* $(ie\hbar/2mc)\,\boldsymbol{\alpha}$*. The magnetic moment is just that assumed in the spinning electron model'* (Pauli model). '*The electric moment, being a pure imaginary, we should not expect to appear in the model. It is doubtful whether the electric moment has any physical meaning.'*

The electron, in addition to the electric charge behaves as though it has some electric and magnetic dipole moments. The magnetic dipole term is the right one to give account of the Zeeman effect in atoms. But, what about the electric dipole? In the last Dirac sentence it is difficult to understand why Dirac, who did not reject the negative energy solutions, and therefore its consideration as the antiparticle states, and insisted that the motion of point $\boldsymbol{r}$ is at the speed of light as an *‘inseparably bound up’* consequence, disliked the existence of this electric dipole which was obtained from his formalism on an equal footing and at the same time as the magnetic dipole term.

In his book and in the Noble dissertation he never mentioned again this electric dipole property. But, what happens if the point $\boldsymbol{r}$ represents the position of the center of charge of the electron. By the previous analysis $\boldsymbol{r}$ seems to be a different point than the center of mass. Then this separation implies that for the center of mass observer there is a nonvanishing electric dipole moment. It is oscillating very fast, its average value is probably zero, but it is a physical property that has to be taken into account.

In fact, in quantum electrodynamics, the complete Dirac Hamiltonian contains both terms. It might happen that this electric dipole does represent the existence of a separation between the center of mass and center of charge.

References

Book

MR, *Kinematical theory of spinning particles*, Fundamental Theories of Physics Series, vol 116, Kluwer, Dordrecht (2001).

Articles

Classical Particle Systems: I. Galilei free particles, J. Phys. **A 18**, 1971 (1985).

Classical Relativistic Spinning Particles, J. Math. Phys. **30**, 318 (1989).

Quantization of generalized spinning particles. New derivation of Dirac's equation, J. Math. Phys. **35**, 3380 (1994).

Are the electron spin and magnetic moment parallel or antiparallel vectors?, LANL ArXiv:physics/0112057.

The dynamical equation of the spinning electron, J. Phys. A, **36**, 4703, (2003), and also LANL ArXiv:physics/0112005.

Classical elementary particles, spin, zitterbewegung and all that. arXiv:physics/0312107.

MR, J.M. Aguirregabiria and A. Hernández, *A pure kinematical explanation of the gyromagnetic ratio $g = 2$ of leptons and charged bosons*, Phys. Lett. **A 257**, 21 (1999).