# A CRITIQUE OF THE ANGULAR MOMENTUM SUM RULES AND A NEW ANGULAR MOMENTUM SUM RULE


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Sum rules, relating the total angular momentum of a nucleon to the spin and orbital angular momentum carried by its constituents, are interesting and important in understanding the internal structure of the nucleon. Indeed it is arguable that the main stimulus for the tremendous present day experimental activity in the field of spin-dependent structure functions was the surprising result of the European Muon Collaborations polarized DIS experiment in 1988, which, via such sum rules, led to what was called a "spin crisis in the parton model", namely the discovery that the spins of its quarks provide a very small contribution to the angular momentum of the proton.

In a much cited paper, Jaffe and Manohar stressed the subtleties involved in deriving general angular momentum sum rules. As they point out, too naive an approach leads immediately to highly ambiguous divergent integrals, and a careful limiting procedure has to be introduced in order to obtain physically meaningful results. In this it is essential to work with non-diagonal matrix elements $\langle p', \sigma | \boldsymbol{J} \ | p \, , \sigma \rangle$ and, this can have some unexpected consequences. Jaffe and Manohar comment that to justify rigorously the steps in such a procedure requires the use of normalizable wave packets, though they do *not* do this explicitly in their paper.

*We show that the results in the literature are incorrect*, and we have taken pains to derive the correct expressions in three different ways, two involving explicit physical wave packets and the third, totally independent, based upon the rotational properties of the state vectors. Surprisingly it turns out that the results are very sensitive to the type of relativistic spin state used to describe the motion of the particle i.e. whether a standard canonical (i.e. boost, as in e.g. Bjorken -Drell) state or a helicity state is utilized.

We present results for the matrix elements of the angular momentum operators, valid in an arbitrary Lorentz frame, both for helicity states and canonical states.

We present a new sum rule for transversely polarized nucleons.

We shall discuss:

1) The origin of the problem

2) The source of errors in the literature

3) Brief summary of our calculations

4) Comparison of our results with those in the literature

5) The new sum rule for transversely polarized nucleons

THE ORIGIN OF THE PROBLEM

In the standard approach one relates the matrix elements of the angular momentum operators to those of the energy-momentum tensor $T^{\mu\nu}(x)$, which is conserved. Typically one deals with expressions like

$$M^{\mu\nu\lambda}(x) \equiv x^{\nu}T^{\mu\lambda}(x) - x^{\lambda}T^{\mu\nu}(x) \qquad (1)$$

The angular momentum operators are space integrals of the spatial components of these, so we would like to know the structure of the forward matrix elements

$$\mathcal{M}^{0ij}(p, \boldsymbol{s}) \equiv \langle p, \boldsymbol{s}| \int d^3x M^{0ij}(\boldsymbol{x}, 0)|p, \boldsymbol{s}\rangle \qquad (2)$$

i.e. what is the functional dependence on the momentum and spin label of the nucleon.

We have:

$$\begin{aligned}\mathcal{M}^{0ij}(p,\boldsymbol{s}) &= \int d^3x \langle p,\boldsymbol{s}|x^i T^{0j}(x) - x^j T^{0i}(x)|p,\boldsymbol{s}\rangle \\ &= \int d^3x\, x^i\, \langle p,\boldsymbol{s}|e^{iP\cdot x} T^{0j}(0) e^{-iP\cdot x}|p,\boldsymbol{s}\rangle \\ -(i\leftrightarrow j) & \\ &= \int d^3x\, x^i\, \langle p,\boldsymbol{s}|T^{0j}|p,\boldsymbol{s}\rangle - (i\leftrightarrow j). \quad (3)\end{aligned}$$

The integral in Eq. (3) is *totally ambiguous*, being either infinite or, by symmetry, zero.

The essential problem is to obtain a sensible physical expression, in terms of $p$ and $\boldsymbol{s}$, for the above matrix element. The fundamental idea is to work with a *non-forward* matrix element and then to try to approach the forward limit. This is similar to what is usually done when dealing with non-normalizable plane wave states and it requires the use of wave packets for a rigorous justification.

THE SOURCE OF ERRORS IN THE LITERATURE

The spin state of the nucleon is labelled by the momentum and the *covariant* spin vector $\mathcal{S}$.

1) The most crucial error in these treatments is the mishandling of the matrix elements of a covariant tensor operator. If $T^{\mu\lambda}$ transforms as a second-rank tensor its *non-forward* matrix elements do *not* transform covariantly. This was the motivation, decades ago, for Stapp to introduce $M$-functions.

Namely, the covariance is spoilt, for canonical spin states by the Wigner rotation, and, for helicity states by the analogous Wick helicity rotation.

Only by first factoring out the wave-functions (in our case Dirac spinors) i.e. by writing

$$\langle p', \mathcal{S}' | T^{\mu\lambda} | p, \mathcal{S} \rangle = \bar{u}(p', \mathcal{S}') \mathcal{T}^{\mu\nu}(p', p) u(p, \mathcal{S}). \tag{4}$$

does the remaining $M$-function, in this case $\mathcal{T}^{\mu\nu}(p', p)$, transform covariantly.

2) For simplicity $\mathcal{S}'$ is chosen equal to $\mathcal{S}$.

A wave packet is constructed

$$|\Psi_{p,\mathcal{S}}\rangle = \frac{N}{\sqrt{(2\pi^3)}} \int d^3q e^{-\lambda^2(\boldsymbol{q}-\boldsymbol{p})^2} |q, \mathcal{S}\rangle \quad (5)$$

But for physical states $q^2 = 0$ and $\mathcal{S}.q = 0$, so for a superposition of physical nucleon states the integration over $\boldsymbol{q}$ is restricted, and this fact is ignored.

The correct way to do it is to build the packet as a superposition of physical plane waves all with the same *rest frame spin vector $\boldsymbol{s}$*

$$|\Psi_{p,\boldsymbol{s}}\rangle = \frac{N}{\sqrt{(2\pi^3)}} \int d^3q e^{-\lambda^2(\boldsymbol{q}-\boldsymbol{p})^2} |q, \boldsymbol{s}\rangle \quad (6)$$

OUR CALCULATIONS

We have used THREE different approaches, all giving the same answer:

1) Relativistic Quantum-Mechanical Dirac particle.

We construct a wave-function corresponding to a superposition of plane wave physical states centered around momentum $\boldsymbol{p}$, all of which have rest-frame spin vector $\boldsymbol{s}$:

$$\psi_{p,\boldsymbol{s}}(\boldsymbol{x},t) = \frac{\bar{N}}{(2\pi)^3}\int \frac{d^3q}{q^0} e^{-\lambda^2(\boldsymbol{q}-\boldsymbol{p})^2} e^{i(\boldsymbol{q}\cdot\boldsymbol{x}-q^0t)} u(q,\boldsymbol{s}), \tag{7}$$

where $q^0 = \sqrt{\boldsymbol{q}^2 + M^2}$, and $u(q,\boldsymbol{s})$ is a standard Dirac spinor.

2) Field Theoretic State

We construct a wave-packet state

$$|\Psi_{p,\boldsymbol{s}}\rangle = \frac{N}{\sqrt{(2\pi^3)}} \int d^3q e^{-\lambda^2(\boldsymbol{q}-\boldsymbol{p})^2} |q, \boldsymbol{s}\rangle \qquad (8)$$

3) A Totally Independent Method Based On The Rotational Properties Of States

This is the simplest most direct approach.

a) It does not need wave packets because it does not use the energy momentum tensor.

b) It works for arbitrary spin, and equally well for *helicity states* or standard *canonical or boost states*.

Let $|p, m\rangle$ be a state with momemtum $p$ which has spin projection $m$ in the rest system.

Under a rotation about axis-$i$ through an angle $\beta$:

$$R_i(\beta)|p, m\rangle = |R_i(\beta)p, n\rangle \mathcal{D}^s_{nm}(R_W(p, \beta)). \quad (9)$$

where $R_W(p, \beta)$ is the Wigner rotation.

Since

$$R_i(\beta) = \exp(-i\beta J_i) \quad (10)$$

it is relatively straightforward to derive the matrix elements of the $J_i$ from Eq. (9).

(For helicity states the Wigner rotation is replaced by the Wick helicity rotation)

COMPARISON OF RESULTS

The nicely covariant looking, but alas wrong, form of the expectation value of the angular momentum operators given by Jaffe-Manohar is:

$$\langle p,\mathcal{S}|\int d^3x M^{0ij}(x)|p,\mathcal{S}\rangle\Big|_{\mathrm{JM}}$$
$$=\frac{1}{4Mp_0}\left[2p^0\epsilon^{ji\beta\sigma}-p^i\epsilon^{0j\beta\sigma}+p^j\epsilon^{0i\beta\sigma}\right]p_\beta\mathcal{S}_\sigma$$

In terms of $\boldsymbol{p}$ and $\boldsymbol{s}$ this leads to

For standard (e.g. Bjorken-Drell) canonical spin states:

$$\langle J_i \rangle_{JM} = \frac{1}{4Mp^0} \left\{ (3p_0^2 - M^2)s_i - \frac{3p_0 + M}{p_0 + M} (\boldsymbol{p} \cdot \boldsymbol{s}) p_i \right\} \quad (11)$$

to be compared with our result

$$\langle J_i \rangle = \tfrac{1}{2} s_i \quad (12)$$

Note: In the above we have left out terms involving the derivative of a delta-function, which vanish for a symmetrical wave-packet.

In general these are different. However, one may easily check that if $\boldsymbol{s} = \widehat{\boldsymbol{p}}$ the Jaffe-Manohar value agrees with Eq.(12), while if $\boldsymbol{s} \perp \widehat{\boldsymbol{p}}$ they are not the same.

Because our result Eq. (12) does not look manifestly covariant, we have shown in detail that it does in fact respect Lorentz invariance.

## SUM RULES

Sum rules, relating the total angular momentum of a nucleon to the spin and orbital angular momentum carried by its constituents, are interesting and important in understanding the internal structure of the nucleon.

In order to deal with the massless gluons we need the analogue of Eq. (12) for helicity states for which we find a surprisingly different result:

$$\langle p', \lambda' | J_i | p, \lambda \rangle = (2\pi)^3 2p_0 \left[ \lambda \eta_i(\boldsymbol{p}) \delta_{\lambda\lambda'} \right] \qquad (13)$$

where

$$\begin{aligned} \eta_x &= \cos(\phi)\tan(\theta/2), \qquad \eta_y = \sin(\phi)\tan(\theta/2), \\ \eta_z &= 1. \end{aligned} \qquad (14)$$

and $(\theta, \phi)$ are the polar angles of $\boldsymbol{p}$ .

The agreement between our results and those of J-M for canonical spin states when $\boldsymbol{s} = \widehat{\boldsymbol{p}}$ is consistent with the much used and intuitive sum rule

$$\tfrac{1}{2} = \tfrac{1}{2}\Delta\Sigma + \Delta G + \langle L^q \rangle + \langle L^G \rangle \qquad (15)$$

In the case that $\boldsymbol{s} \perp \widehat{\boldsymbol{p}}$ we find a new sum rule. For a proton with transverse spin vector $\boldsymbol{s}_T$ we find

$$\tfrac{1}{2} = \tfrac{1}{2} \sum_{q,\,\bar{q}} \int dx\, \Delta_T q^a(x) + \sum_{q,\,\bar{q},\,G} \langle L_{\boldsymbol{s}_T} \rangle^a \qquad (16)$$

where $L_{\boldsymbol{s}_T}$ is the component of $\boldsymbol{L}$ along $\boldsymbol{s}_T$. The structure functions $\Delta_T q^a(x) \equiv h_1^q(x)$ are known as the quark transversity or transverse spin distributions in the nucleon. Note that no such parton model sum rule is possible with the Jaffe-Manohar formula because, as $p \to \infty$, Eq. (11) for $i = x, y$ diverges.

The result Eq. (16) has a very intuitive appearance, very similar to Eq. (15).

The structure functions $\Delta_T q^a(x) \equiv h_1^q(x)$ are most directly measured in doubly polarized Drell-Yan reactions where the asymmetry is proportional to

$$\sum_a e_a^2 [\Delta_T q^a(x_1) \Delta_T \bar{q}^a(x_2) + (1 \leftrightarrow 2)]. \quad (17)$$

They can also be determined from the asymmetry in semi-inclusive hadronic interactions like

$$p + p(\boldsymbol{s}_T) \to H + X$$

where $H$ is a detected hadron, typically a pion, and in SIDIS reactions with a transversely polarized target

$$\ell + p(\boldsymbol{s}_T) \to \ell + H + X.$$

The problem is that in these semi-inclusive reactions $\Delta_T q^a(x)$ always occurs multiplied by the largely unknown Collins fragmentation function.

SUMMARY

1) The standard derivation of the tensorial structure of the expectation value of the angular momentum $\boldsymbol{J}$, for a relativistic spin-$s$ particle, in which the matrix elements of the angular momentum operators are related to the matrix elements of the energy-momentum tensor, is rendered difficult by the singular nature of the operators involved.

We have shown that the results in the literature are incorrect, and have derived the correct expressions in three different ways, two of them based on a careful wave-packet treatment of the standard approach, and the third, quite independent, based on the known rotational properties of the spin states, which circumvents the use of the energy-momentum tensor. All three methods yield the same results.

2) We have shown that, surprisingly, the results for helicity states are very different from those for standard canonical spin states.

3) Using a Fock-space picture of the proton, we have used our results to obtain a new sum rule for a transversely polarized nucleon, which involves the transverse spin or transversity distribution $\Delta_T q(x) \equiv h_1(x)$, and which is similar in form to the classic longitudinal spin sum rule.